Classification on the average of random walks

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Abstract. We introduce a new method for studying large scale properties of random walks. The new concepts of *transience* and *recurrence on the average* are compared with the ones introduced in [1] and with the usual ones; their relationships are analyzed and various examples are provided.

Keywords: Random walk, limit on the average, generating function, summability methods.

Mathematics Subject Classification: 60G50, 82B41

1. Introduction

Random walks on graphs provide a mathematical model in many scientific areas, from finance (financial modelling), to physics (magnetization properties of metals, evolution of gases) and biology (neural networks, disease spreading). In particular graphs describe the microscopical structure of solids, ranging from very regular structures like crystals or ferromagnetic metals which are viewed as Euclidean lattices, to the irregular structure of glasses, polymers or biological objects.

Geometrical and physical properties of these discrete structures are linked by random walks (especially the simple random walk), which usually describe the diffusion of a particle in these more or less regular media.

An interesting feature of random walks on graphs is their large time scale asymptotics which is deeply connected with the concept of recurrent or transient random walk. This classification was first introduced by Pólya for simple random walks on lattices (see [2]) to distinguish between random walks which return to the starting point with probability one (these are recurrent), and those whose return probability is less than one (which are transient).

We observe that in a vertex-transitive graph (such as the lattice $\mathbb{Z}^d$) the return probabilities of the simple random walk do not depend on the starting vertex; but in the case of

any irreducible random walk they may differ from vertex to vertex, although being strictly less than one in one vertex is equivalent to being strictly less than one in any vertex. The distinction between recurrent and transient random walks is known as the type-problem (for the type-problem for random walks on infinite graphs, see [3]).

It has been recently observed that even though the type of a random walk describes local properties of the physical model, average values of return probabilities over all starting sites play a key role in the comprehension of the macroscopical behaviour of the model itself (like spontaneous breaking of continuous symmetries [4], critical exponents of the spherical model [5], or harmonic vibrational spectra [6]).

These observations lead to the definition of a new type-problem: the type-problem on the average (see [1]). The definition introduced by Burioni, Cassi and Vezzani is the following: given the family of the generating functions of the n-step return probabilities of a random walk, $\{F(x,x|z)\}_{x \in X}$, and a reference vertex $o \in X$ (where $(X, E(X))$ is the graph to which the random walk is adapted), the random walk is recurrent on the average if

$$\lim_{z \to 1^-} \lim_{n \to \infty} \frac{\sum_{x \in B(o,n)} F(x,x|z)}{|B(o,n)|} = 1, \qquad (1)$$

and transient on the average if the value of the double limit is less than 1 ($B(o,n)$ is the closed ball of center o and radius n, $|\cdot|$ denotes cardinality).

The "average" mentioned in the name given to this new type-problem is a repeated average over balls with fixed center and increasing radii (of course existence of the limit of these averages is implicitly required). This procedure is a particular case of the following: given a sequence $\{\lambda_n\}_n$ of probability measures on the set X, for each n we consider the average of F with respect to λ_n (that is the expected value of F with respect to λ_n) and then we take the limit of these averages when n goes to infinity. Note that in definition (1) one has to evaluate a further limit (namely the one for z going to 1) and $\lambda_n(x) = \chi_{B(o,n)}(x)/|B(o,n)|$.

From a mathematical point of view the definition of this "limit on the average" leads to some problems, like the existence of the limit, the possibility of exchanging the order of the two limits and the dependence on the reference vertex o.

We provide an example of random walk which has no classification on the average in the above sense (the simple random walk on a bihomogeneous tree). Thus the classification on the average is not complete, while the classical one in recurrent and transient random walks is complete (we will refer to the usual classification as the "local" one, in contrast with the one "on the average").

We then propose a new classification on the average which is complete and is in many cases an extension of the one given in [1]. For this new definition we analyze its independence on the reference vertex and provide a sufficient condition which is weaker than the one produced in [1]. We make comparisons between the former and the new definitions of classification on the average and with the local one; we study when these definitions agree and we give examples of random walks which behave differently according to different classifications (that is, which are transient with respect to one of these classifications and recurrent according to another one).

Another question which naturally arises is what can be said when averages are taken

over general sets (not necessarily balls), that is when $\{\lambda_n\}_n$ is defined as $\{\chi_{B_n}/|B_n|\}_n$ where $\{B_n\}_n$ is an increasing family of subsets. Moreover $\{\lambda_n\}_n$ could be a general family of probability measures (for instance, for some reasons one would like to give to some subgraphs a greater weight than the one given to other subgraphs). We deal with these more general averaging procedure and prove results which generalize the particular cases.

We briefly outline the content of the paper. In Section 2 we define the *limit on the average*, we recall the distinction between *thermodynamical transience and recurrence on the average* ($\mathrm{TOA_t}$ and $\mathrm{ROA_t}$) as defined in [1].

In Section 3 we show that the simple random walk on the bihomogeneous tree has no thermodynamical classification (Examples 3.1 and 3.11) and we introduce our classification on the average of random walks (TOA and ROA). The rest of the section is devoted to the study of averages over balls: we prove that under certain conditions the classification is independent of the centre of the balls (Proposition 3.3). We compare the classical concepts of recurrence and transience with the corresponding "on the average" and the "thermodynamical" ones (Theorem 3.5 and Examples 3.7 and 3.8, see also Table 1). A flow criterion on the average is stated (Theorem 3.10), which should be compared with the "classical" one of Lyons [7] and Yamasaki [8].

In Section 4 we connect the behaviour of the random walk on the subgraph to the behaviour of the random walk on the whole graph. Corollary 4.4 and Theorem 4.6 deal with the classification on the average, Theorem 4.10 with the thermodynamical one.

Sections 5 and 6 are devoted respectively to averages over families of finite sets and general averages. The two appendices present technical results for averages of general functions and families of power series.

Table 1: Comparison between the three classifications

	$\mathrm{TOA}, \mathrm{TOA_t}$	$\mathrm{TOA}, \mathrm{ROA_t}$	$\mathrm{ROA}, \mathrm{TOA_t}$	$\mathrm{ROA}, \mathrm{ROA_t}$
Recurrent	impossible (Th. 6.2(ii))	impossible (Th. 6.2(ii))	Ex. 5.4	$\mathbb{Z}^2$
Transient	$\mathbb{Z}^3$	impossible (Th. 6.2(iii))	Ex. 3.7	Ex. 3.8

2. Basic definitions

We start giving the general definition of a large scale average depending on a sequence of probability measures on an at most countable set X (we will usually think of X as the vertex set of an infinite, connected and locally finite graph).

Definition 2.1. *Let $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on X; we call* limit on the λ-average *(or , if there is no ambiguity,* limit on the average*) the linear map*

$$L_\lambda(f) := \lim_{n\to+\infty} \sum_{x\in X} f(x)\lambda_n(x).$$

We call $\mathcal{D}(L_\lambda)$ the domain of L_λ, that is

$$\mathcal{D}(L_\lambda) := \Big\{ f \in \mathbb{C}^X : \sum_{x\in X} |f(x)|\lambda_n(x) < +\infty, \forall n \in \mathbb{N}, \\ \textit{and } \exists \lim_{n\to+\infty} \sum_{x\in X} f(x)\lambda_n(x). \Big\}$$

*If $A \subseteq X$ is such that $\chi_A \in \mathcal{D}(L_\lambda)$, then A is called L_λ-*measurable *(or briefly* measurable*) and with a slight abuse of notation, we write $L_\lambda(A)$ instead of $L_\lambda(\chi_A)$ (and we call it the L_λ-measure of A or simply the measure of A).*

If $\mathcal{F} = \{B_n\}_{n\in\mathbb{N}}$ is an increasing family of finite subsets whose union is X, we call *limit on the average* with respect to $\mathcal{F}$ (we denote it by $L_\mathcal{F}$) the limit on the λ-average where $\lambda_n(x) = \chi_{B_n}(x)/|B_n|$.

When X is a metric space (in our case a locally finite, non-oriented graph with its natural distance) and $o \in X$, we study the *limit on the average* where $\lambda_n(x) = \chi_{B(o,n)}(x)/|B(o,n)|$, and we will write L_o instead of L_λ.

The limits on the average are particular cases of *summability methods* (see for instance [9] Paragraph 4.10); if $\lim_{n\to\infty} \lambda_n(x) = 0$ for any $x \in X$ (i.e. every finite subset of X is measurable and its measure is zero) then the limit on the average is called *regular*.

We want now to point out some remarks on the Definition 2.1.

Remark 2.2. Given a limit on the average on X, the set of measurable subsets is not, in general, a σ-algebra (nor an algebra: see Proposition 5.2). Anyway it is easy to show that: (i) if S is a measurable set such that $L_\lambda(S) = 0$ and $S' \subset S$ then S' is measurable too and $L_\lambda(S') = 0$; (ii) if A is measurable and its measure is 0 then for every bounded complex function $f : X \to \mathbb{C}$, we have that $\chi_A f \in \mathcal{D}(L_\lambda)$ and $L_\lambda(\chi_A f) = 0$.

Remark 2.3. We defined L_λ on complex valued functions mostly for technical reasons (integrals in the complex field will be needed), nevertheless we are interested in real valued functions. In particular functions taking possibly the values $\pm\infty$ should be admitted (take for instance f equal to the Green function of a random walk, which we will define in a moment). To this aim let us consider a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, such that, for all $n \in \mathbb{N}$, at least one of the following conditions holds:

$$\begin{cases} \sum_{x\in X: f(x)>0} f(x)\lambda_n(x) < +\infty \\ \sum_{x\in X: f(x)<0} f(x)\lambda_n(x) > -\infty. \end{cases} \tag{2}$$

For every such function we introduce the *upper limit on the λ-average* and *lower limit on the λ-average* as

$$\sup L_\lambda(f) := \limsup_{n\to+\infty} \sum_{x\in X} f(x)\lambda_n(x),$$

$$\inf L_\lambda(f) := \liminf_{n\to+\infty} \sum_{x\in X} f(x)\lambda_n(x).$$

We easily note that if f is any real valued function satisfying the above condition then $f \in \mathcal{D}(L_\lambda)$ if and only if $\inf L_\lambda(f) = \sup L_\lambda(f) \in \mathbb{R}$.

Remark 2.4. Since any bounded function (hence any characteristic function of a subset of X) satisfies both equations in (2), we have that every subset is $\inf L_\lambda$-measurable and $\sup L_\lambda$-measurable (note that these "measures" are not even finitely additive, although they are defined on $\mathcal{P}(X)$). Also note that for every $A \subseteq X$, being A L_λ-measurable with measure 1 is equivalent to $\inf L_\lambda(A) = 1$ (equivalently, A^c is L_λ-measurable with zero measure).

Now we define the functions related to random walks, of which we will consider averages through the main part of this paper.

Given a given random walk (X, P) we denote by $p^{(n)}(x, y)$ the n-step transition probabilities from x to y ($n \geq 0$) and by $f^{(n)}(x, y)$ the probability that the random walk starting from x hits y for the first time after n steps ($n \geq 1$). Then we define the Green function $G(x, y|z) = \sum_{n\geq 0} p^{(n)}(x, y)z^n$ and the generating function of the first time return probabilities $F(x, y|z) = \sum_{n\geq 1} f^{(n)}(x, y)z^n$ where $x, y \in X$, $z \in \mathbb{C}$ (further details can be found in [3] Chapter 1.B, where F is called U).

An irreducible random walk (X, P) is recurrent if $F(x, x) := F(x, x|1) = 1$ for some $x \in X$ (equivalently for all x) and transient if $F(x, x) < 1$ for some $x \in X$ (equivalently for all x).

We recall here the flow criterion which characterizes transient networks. One can associate an electric network to a reversible random walk (X, P) with reversibility measure m in the following way. We endow any edge with an orientation $e = (e^-, e^+)$ and with a resistance $r(e) = (m(e^-)p(e^-, e^+))^{-1}$ (in the case of the simple random walk $r(e) = 1$ for every edge e).

A flow u from a vertex x to infinity with input i_0 is a function defined on $E(X)$ such that

$$\sum_{e:e^-=y} u(e) = \sum_{e:e^+=y} u(e) + i_0\delta_x(y), \qquad \forall y \in X.$$

The energy of u is defined as $<u, u>:= \sum_{e\in E(X)} u(e)^2 r(e)$. The existence of finite energy flows is related with transience by the following theorem (here $\mathrm{cap}(x)$ is the capacity of the set $\{x\}$: we refer to [3] for the definition).

Theorem 2.5. *Let (X, P) be a reversible random walk. The following are equivalent:*

(a) the random walk is (locally) transient;

(b) there exists $x \in X$ (equivalently for all $x \in X$) such that it is possible to find a finite energy flow with non-zero input, from x to infinity;

(c) there exists $x \in X$ (equivalently for all $x \in X$) such that $\mathrm{cap}(x) > 0$.

We restate the definition of the type-problem according to [1] (and we call it "thermodynamical" to distinguish it from the definition which will be given later).

Definition 2.6. *Let (X, P) be a random walk, and $o \in X$ a fixed vertex. Suppose that $F(\cdot, \cdot|z) \in \mathcal{D}(L_o)$, for all $z \in (\varepsilon, 1)$, for some $\varepsilon \in (0, 1)$. The random walk is said* thermodynamically transient on the average *with respect to o (briefly* $\mathrm{TOA_t}$*) if*

$$\lim_{z\to 1^-} L_o(F(z)) < 1, \tag{3}$$

(where $L_o(F(z)) := L_o(F(\cdot,\cdot|z))$), thermodynamically recurrent on the average with respect to o (ROA$_\mathrm{t}$) *if the limit is equal to* 1*.*

3. The classification on the average (over balls)

From now on, if not otherwise stated, we will assume that $(X, E(X))$ is a connected (infinite), locally finite, non oriented graph, that o is a fixed vertex of X, that (X, P) is a random walk, not necessarily adapted to the graph $(X, E(X))$, and that the limit on the average is L_o.

Some natural question are: is any random walk either TOA$_\mathrm{t}$ or ROA$_\mathrm{t}$ (that is, is the classification on the average complete)? Does the classification depend on the choice of o? Can we reverse the order of the two limits in equation (3)?

Regarding the first question, it is not difficult to find examples of random walks with no thermodynamical classification.

Example 3.1. Let us consider the class of bihomogeneous trees (which coincides with the class of trees which are radial with respect to every point, see [10] Proposition 2.9). Despite its property of symmetry, the simple random walk on a bihomogeneous tree $\mathbb{T}_{n,m}$ (with $n \neq m$) is neither ROA$_\mathrm{t}$ nor TOA$_\mathrm{t}$ (for the proof, see Example 3.11).

It would be desirable that the classification on the average would not depend on the choice of the reference vertex o. It has been shown in [1] Section 4 that, if the graph has bounded geometry and

$$\lim_{n\to+\infty} \frac{|\partial B(o,n)|}{|B(o,n)|} = 0 \tag{4}$$

(where $\partial B(o,n) := \{x \in B(o,n) : \exists y \notin B(o,n),\, (x,y) \in E(X)\}$) for some o, then the limit on the average is independent of the choice of o. This condition is not satisfied, for instance, by any homogeneous tree of degree greater than 2, or by any "fast growing" graph.

As for the last question, that is whether the limit in equation (3) coincides with $L_o(F)$, in general the answer is no. Anyway, exploiting the fact that F is a power series with non negative coefficients one can show that at least when $\sum_{n\geq 1} k_n$ converges, where $k_n = \sup_{x\in X} f^{(n)}(x,x)$, then existence of the limit in (3) implies existence of $L_o(F)$ and these limits coincide (see Proposition 3.5 (iii) and (iv)).

In order to overcome these difficulties, we introduce a new classification on the average.

Definition 3.2. *Let (X, P) a random walk, and $\{\lambda_n\}_n$ a sequence of probabilities measures on X, the random walk is called* transient on the average with respect to λ (λ-TOA) *if*

$$\inf L_\lambda(F) := \liminf_{n\to\infty} \sum_{x\in X} F(x,x)\lambda_n(x) < 1,$$

recurrent on the average with respect to λ (λ-ROA) *if the limit is equal to* 1*.*

Since this limit always exists, this classification is complete (that is, any random walk is either λ-TOA or λ-ROA).

In this section, if not otherwise stated, $\lambda_n = \chi_{B(o,n)}/|B(o,n)|$ (we consider classification with $\inf L_o$) and we write TOA and ROA instead of λ-TOA and λ-ROA.

Now we exhibit a condition implying that this new classification (which in the rest of this paper we denote by "the classification on the average", in contrast with the "thermodynamical" one defined by (3)) does not depend on the fixed vertex o. As in the thermodynamical case, this condition is a topological one for the underlying graph.

Proposition 3.3. *Let $(X, E(X))$ be such that there exists $x \in X$ satisfying*

$$\sup_{n\in\mathbb{N}} \frac{|S(x,n+1)|}{|B(x,n)|} < +\infty, \tag{5}$$

(where $S(x, n+1)$ is the sphere centered in x with radius $n+1$) then the classification on the average of any random walk is independent of the choice of o.

Proof. We note that equation (5) holds for some x if and only if it holds for any vertex of X. It is easy to show that (5) is equivalent, in the case of the $\inf L_o$ classification, to the requests of Proposition 6.1 (i). □

This condition is weaker than the one for the thermodynamical classification (equation (4)); indeed observe that (5) is satisfied by any graph with bounded geometry. On the other hand, bounded geometry is not necessary, as is shown by the following example.

Example 3.4. Given a strictly increasing sequence of natural numbers $\{s_j\}_j$, such that $s_0 \geq 1$, and a vertex x_0, construct the tree T as follows (see Figure 1, where $s_j = j$). Each element on the sphere $S(x_0, m)$ has exactly one neighbour on the sphere $S(x_0, m+1)$ if $m \neq s_j$ for any $j \in \mathbb{N}$ and exactly j neighbours if $m = s_j$. If we choose $s_{j+1} \geq s_j + j + 1$ then T satisfies equation (5) and has not bounded geometry.

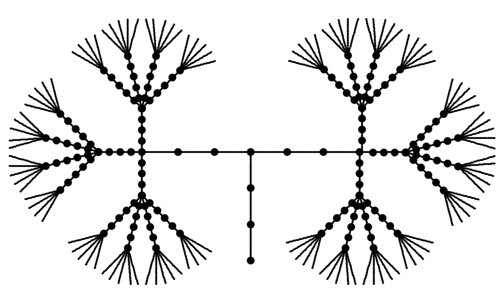

Figure 1

Now we start comparing the two classifications on the average and the local one.

Proposition 3.5. *Let (X, P) be a random walk and let ∞ be the point added to X in order to construct its one point compactification.*

(i) If there exists $A \subseteq X$ measurable, such that $L_o(A) = 1$ and $\lim_{x\to\infty, x\in A} F(x,x) = \alpha$ then $L_o(F)$ exists, and is equal to α. Thus the random walk is TOA *(respectively* ROA*) if and only if $\alpha < 1$ (respectively $\alpha = 1$);*

(ii) if (X, P) is (locally) recurrent then $L_o(F)$ exists, is equal to 1 *and the random walk is* ROA*;*

(iii) *if (X,P) is* $\mathrm{ROA_t}$ *then $L_o(F)$ exists, is equal to* 1 *and the random walk is* ROA*;*
(iv) *if the series $F(x,x)$ is totally convergent (with respect to $x \in X$) and (X,P) is* $\mathrm{TOA_t}$ *then $L_o(F)$ exists, it is less than* 1 *and the random walk is* TOA*;*
(v) *(X,P) is* ROA $\Longleftrightarrow$ *for every $\varepsilon > 0$ the set $\{x : F(x,x) \geq 1-\varepsilon\}$ is measurable with measure* 1*;*
(vi) *(X,P) is* ROA $\Longleftrightarrow$ *there exists $A \subseteq X$ measurable, such that $L_o(A) = 1$ and $\lim_{\substack{x\to\infty \\ x\in A}} F(x,x) = 1$;*
(vii) *(X,P) is* TOA $\Longleftrightarrow$ *there exists $A \subseteq X$ such that* $\sup L_o(A) > 0$ *and* $\sup_A F(x,x) < 1$.

The proof is a particular case of the proof of Proposition 6.2.

Proposition 3.5(iv) states that being the series $F(x,x)$ totally convergent guarantees that the classification on the average and the thermodynamical one agree (if the last one is admissible). Obviously the function $F(x,x)$ needs not to be totally convergent even in the case of simple random walks (see Examples 3.7 and 5.4).

Under certain conditions, the series $F(x,x)$ is indeed totally convergent.

Proposition 3.6. *Let (X,P) be a random walk adapted to the graph $(X,E(X))$. If one of the following conditions holds then the series $F(x,x)$ is totally convergent (with respect to x).*
(i) *There exists a subset Γ of $\mathrm{AUT}(X)$ (the automorphism group of the graph) and a finite subset $X_0 \subset X$ with the property that for any $y \in X$ there exist $x \in X_0$ and $\gamma \in \Gamma$ such that $\gamma(x) = y$ and P is Γ-invariant.*
(ii) *The radius of convergence of the Green function $G(x,x|z)$ (which is independent of x) is $r > 1$.*
(iii) *(X,P) is reversible (with reversibility measure m and total conductance $a(x,y) := m(x)p(x,y)$) and it satisfies the strong isoperimetric inequality that is*

$$\sup_{A\subset X} \frac{m(A)}{s(A)} < +\infty,$$

where the supremum is taken over finite subsets A and $s(A) := \sum_{x\in A, y\in A^c} a(x,y)$.

Proof. We just outline the main points.
(i) If $y = \gamma(x)$ and P is invariant under the action of γ then $f^{(n)}(x,x) = f^{(n)}(y,y)$. By hypotheses $k_n := \sup_{x\in X} f^{(n)}(x,x) = \max_{x\in X_0} f^{(n)}(x,x) \leq \sum_{x\in X_0} f^{(n)}(x,x)$. Hence $\sum_{n=0}^\infty k_n \leq \sum_{x\in X_0} F(x,x) \leq |X_0|$.
(ii) It follows from $f^{(n)}(x,x) \leq p^{(n)}(x,x) \leq 1/r^n$ which holds for every $x \in X$ and every $n \in \mathbb{N}$.
(iii) See [3] Chapter 2 Theorems 10.3 and 10.9 and apply (ii). $\square$

For instance (i) applies to Γ-invariant random walks, where Γ is a subgroup and X has a finite number of orbits with respect to Γ. This is the case of random walks adapted to Cayley graphs or of the simple random walk on quasi transitive graphs.

As for condition (ii), an example is given by a locally finite tree with minimum degree 2 and with finite upper bound to the lenghts of its unbranched geodesics.

We observe that even if (X, P) is both thermodynamically classifiable and classifiable on the average, the two classifications may not agree, as is shown by the following example.

Example 3.7. Let $X := \bigcup_{n \in \mathbb{N}} \{n\} \times \mathbb{Z}_{n+1}$. For any $n, m \in \mathbb{N}$, $p \in \mathbb{Z}_{n+1}, q \in \mathbb{Z}_{m+1}$, (n, p) and (m, q) are neighbours if and only if one of the following holds (see Figure 2)

1) $p = 0_{\mathbb{Z}_{n+1}}$ and $q = 0_{\mathbb{Z}_{m+1}}$ and $|m - n| = 1$,
2) $m = n$ and $p - q = \pm 1$, (where $p - q$ is the usual operation in $\mathbb{Z}_{n+1}$).

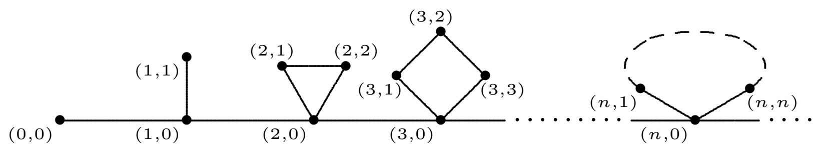

Figure 2

If $\{p_n\}$ is a $(0, 1)$-valued sequence such that $p_n^n \uparrow 1$ and $\alpha \in \mathbb{R}$, $\alpha < 1/3$, then we define the (adapted) transition probabilities as follows:

$$\begin{array}{rll} p((0,0),(1,0)) = & p((1,1),(1,0)) := 1, & \\ p((1,0),(1,1)) := & p_1 + (1-p_1)\alpha, & \\ p((n,0),(n-1,0)) := & (1-p_n)\alpha, & n \geq 1, \\ p((n,0),(n+1,0)) := & (1-p_n)(1-2\alpha), & n \geq 1, \\ p((n,p),(n,p+1)) := & p_n, & n \geq 2, \\ p((n,p),(n,p-1)) := & (1-p_n), & n \geq 2, p \neq 0, \\ p((n,0),(n,n-1)) := & (1-p_n)\alpha, & n \geq 2. \end{array}$$

By using standard stopping time arguments we easily see that this random walk is locally transient.
If we denote by $C_n := \{(n,p) : p \in \mathbb{Z}_{n+1}\}$ for every $n \in \mathbb{N}$, hence for any $x \in C_n$, we have that $f^{(n)}(x,x) \geq p_n^n$ and $f^{(m)}(x,x) \leq 1 - f^{(n)}(x,x)$ for all $m \neq n$. Thus $\lim_{x\to\infty} f^{(m)}(x,x) = 0$ for any $m \in \mathbb{N}$ and if $z \in (0,1)$ by Bounded Convergence Theorem (using $z^m \geq f^{(m)}(x,x)z^m$) we derive $\lim_{x\to\infty} F(x,x|z) = 0$. Whence for any regular λ we obtain $L_\lambda(F(z)) := L_\lambda(F(\cdot,\cdot|z)) = 0$ which implies that the random walk is λ-TOA$_\mathrm{t}$.
On the other hand $F(x,x) \geq f^{(m)}(x,x)$ for any $x \in X$, $m \in \mathbb{N}$, hence if $x \in \cup_{m\geq n} C_m$ we have that $F(x,x) \geq \inf_{m\geq n} p_m^m = p_n^n$ which implies $\lim_{x\to\infty} F(x,x) = 1$ and, for any regular λ, $L_\lambda(F) = 1$ (that is, the random walk is λ-ROA). Since the classification on the average and the thermodynamical one are different this provides an example of a random walk for which the series $F(x,x)$ is not totally convergent (Proposition 3.5(iv)).

Let us now make some comparisons between the local classification and the classification on the average of a random walk. The previous example, which is locally transient, TOA$_\mathrm{t}$ and ROA, shows also that while local recurrence imply recurrence on the average, local transience does not imply transience on the average. There are also examples of locally transient, ROA$_\mathrm{t}$ and ROA random walks, as is shown by the following.

Example 3.8. Given the sequence of natural numbers $\{s_j = \sum_{i=1}^{j} \beta^i\}_{j\geq 1}$, where $\beta \geq 2$ is an integer number, $s_0 = 0$ and o is a vertex, the construction of the tree T is

similar to the one in Example 3.4. Each element on the sphere $S(o,m)$ has exactly one neighbour on the sphere $S(o,m+1)$ if $m \neq s_j$ for any $j \geq 0$ and exactly α neighbours if $m = s_j$ ($\alpha \in \mathbb{N}$) (Figure 3 represents the case $\alpha = 3$, $\beta = 2$). An application of Theorem 2.5 proves that T is locally transient if and only if $\alpha > \beta$ (see for instance [10] Remark 4.3).

It is easy to prove that the set A obtained by removing from X the balls of radius k centered in the elements of $S(o, s_k)$, for all $k \in \mathbb{N}$, has L_o-measure equal to 1. Moreover on A, for every fixed n, as x tends to infinity $f^{(n)}(x,x)$ is definitively equal to $f^{(n)}_{\mathbb{Z}}(0,0)$ (the first time return probabilities of the simple random walk on $\mathbb{Z}$). Hence by Proposition 3.5(i) the graph is $\mathrm{ROA_t}$ (thus ROA) with respect to any reference vertex.

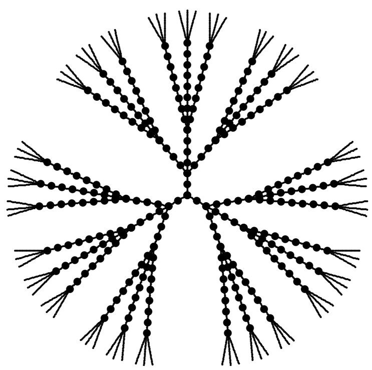

Figure 3

It is known that (local) transience is equivalently expressed by one of the following conditions: (i) $G(x,x) := G(x,x|1) = +\infty$ for some (i.e. for every) $x \in X$; (ii) $F(x,x) = 1$ for some (i.e. for every) $x \in X$. In the average case we can only claim a partial result.

Proposition 3.9. *Let (X,P) be a random walk,. Then:*
(i) if the random walk is $\mathrm{ROA_t}$ *then* $\lim_{z \to 1^-} \inf L_o(G(z)) = +\infty$*;*
(ii) if the random walk is ROA *then* $L_o(G) = +\infty$.

For the proof we refer to the general case, see Proposition 6.5. Observe that in Proposition 3.9 (i) existence of $L_o(G(z))$ is not guaranteed and then we have to consider $\inf L_o$ instead. Also notice that reversed implications are not true, see for instance Example 5.3 (according to [1] this is an example of a *mixed* $\mathrm{TOA_t}$ graph).

Theorem 2.5 gives a useful tool to (locally) classify reversible random walks. A similar result can be stated for the classification on the average.

Theorem 3.10. *Let (X,P) be a reversible random walk, with reversibility measure m satisfying* $\inf m(x) > 0$, $\sup m(x) < +\infty$ *(in particular this condition is satisfied by the simple random walk on a graph with bounded geometry). Then TFAE:*
(a) the random walk is TOA*;*
(b) there exists $A \subseteq X$ such that $\sup L_o(A) > 0$, *there is a finite energy flow u^x from x to ∞ with non-zero input i_0 for every $x \in A$ and* $\sup_{x\in A} < u^x, u^x > < +\infty$*;*
(c) there exists $A \subseteq X$ such that $\sup L_o(A) > 0$ *and* $\inf_{x \in A} \mathrm{cap}(x) > 0$.

For the proof see Theorem 6.3.

As an application we classify bihomogeneous trees and a whole family of inhomogeneous trees.

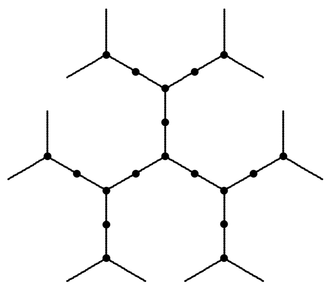

Figure 4

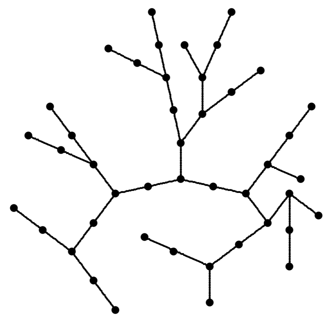

Figure 5

Example 3.11. Consider the bihomogeneous tree $\mathbb{T}_{m,n}$ and a couple of vertices x_n and x_m, the first with degree n and the second with degree m (see Figure 4 for the case $m = 3$ and $n = 2$). We can construct two finite energy flows u^n and u^m with fixed input i_0, respectively from x_n to infinity and from x_m to infinity. But then we can obtain a finite energy flux from any vertex x to infinity (with input i_0) by translating u^n or u^m (depending on the degree of x). Thus we can construct a family of fluxes with bounded energy and this proves that the simple random walk on $\mathbb{T}_{m,n}$ is TOA. The proof, which is based on the ideas of Theorem 4.6, can be repeated for any λ. Moreover, since $L_o(F)$ does not exist for any reference vertex o, and the series $F(x, x)$ is totally convergent by Proposition 3.6(i) (with $X_0 = \{x_n, x_m\}$ and $\Gamma = \mathrm{AUT}(X)$) then by Proposition 3.5(iii) and (iv) the simple random walk on $\mathbb{T}_{m,n}$ cannot be thermodynamically classifiable.
Analogously one can show that the simple random walk on a tree $T'_{k,n}$ whose vertices have degree 2 or k ($k \geq 3$) and such that the distance between ramifications is n ($n \geq 2$) is TOA (while in the former case we had essentially only two fluxes, here we have at most $[n/2] + 1$ fluxes).
Now consider an inhomogeneous tree $T''_{k,n}$ whose vertices have degree 2 or k ($k \geq 3$) and such that the distance between ramifications does not exceed n ($n \geq 2$): see Figure 5 for the case $k = 3$ and $n = 2$. Here the family of finite energy fluxes with fixed input, constructed from any vertex to infinity, is in general infinite, but the supremum of the energy is bounded by the supremum of the energy of the fluxes on $T'_{k,n}$. Hence the simple random walk on $T''_{k,n}$ is TOA.

As we have seen, the family of L_o-measurable sets plays an important role in the classification on the average of random walks, but one has to be careful when dealing with such sets, since they are not an algebra. We state the following proposition, as for the proof, see Proposition 5.2.

Proposition 3.12. *Let $o \in X$. The class of L_o-measurable subsets is not an algebra; in particular there exist two measurable subsets of X, A and B, such that $A \cap B$ is not measurable.*

4. Subgraphs and graphs

In this section we study information which can be inferred from the knowledge of the behaviour of random walks on subgraphs.

Since we have to average on a subgraph, the first thing to do is to rescale the weights. We start in a general setting, where X is an at most countable set, and we average a general function f.

Definition 4.1. *Let $\{\lambda_n\}_n$ be a sequence of probability measures on X and $S \subseteq X$ such that $\lambda_n(S) > 0$, for all $n \in \mathbb{N}$. Then the limit on the average L^S_λ defined on S by $\lambda^S_n := \lambda_n|_S/\lambda_n(S)$ for every $n \in \mathbb{N}$ and $x \in S$ is called* rescaled limit on the average*.*

As usual, L^S_o will be the limit in the case of the average over balls. The following proposition links L^S_λ and L_λ.

Proposition 4.2. *Let $S \subseteq X$ be an L_λ-measurable subset with positive L_λ-measure. If $f \in \mathbb{C}^X$, then:*

(i) $f|_S \in \mathcal{D}(L^S_\lambda) \iff \chi_S \cdot f \in \mathcal{D}(L_\lambda)$,

(ii) if $f|_S \in \mathcal{D}(L^S_\lambda)$ *then* $L_\lambda(\chi_S \cdot f) = L^S_\lambda(f|_S) \cdot L_\lambda(S)$.

(iii) $\inf L_\lambda(\chi_S \cdot f) = \inf L^S_\lambda(f|_S) \cdot L_\lambda(S)$.

Proof. The proof is straightforward and we omit it. □

From now on, we consider the averaging process over balls of the generating function F related to a random walk (X, P). There are two different ways of looking at the behaviour of the random walk on a subgraph $S \subseteq X$. The first one is to consider S as a subset of the graph with $F(x, x)$ restricted to the sites in S. The second approach is to view S as an independent graph, with possibly different generating functions $F(x, x)$.

We start with the first point of view. Proposition 4.2 implies that sets of measure zero have no weight in the averaging procedure (think of S such that $L_\lambda(S) = 1$: then $L^S_\lambda(F|_S) = L_\lambda(F)$). In particular, if we can find two subsets such that one of the two grows strictly slowlier than the other one, then the first one can be neglected in the averaging process.

Remark 4.3. Let $X = X_1 \cup X_2$, where $X_1 \cap X_2$ is finite. Suppose that $|B(o, n) \cap X_1| \le f(n)$, $|B(o, n) \cap X_2| \ge g(n)$ for all $n \in \mathbb{N}$, where f and g are two functions such that $f(n)/g(n)$ tends to zero as n goes to infinity. Then $L_o(X_1) = 0$ (hence the classification of any random walk on X depends only on the restriction of the generating function F on X_2).

The following corollary of Proposition 4.2 links classification on a subgraph with classification on the whole graph.

Corollary 4.4. *Let (X, P) be a random walk and let S be a subgraph of X such that $L_o(S) > 0$. If the restriction of F to the subgraph satisfies $L^S_o(F|_S) < 1$ then (X, P) is* TOA*.*

Another result which links the behaviour of F on subsets with the behaviour of F on

the whole graph is the following (note that this result holds for any λ and any function f in place of F).

Proposition 4.5. *Let $\overline{X} := X \cup \{\infty\}$ be the one point compactification of X with the discrete topology and let $\{A_i\}_{i\in\mathbb{N}}$ be a partition of X such that A_i is L_o-measurable and for every i such that $L_o(A_i) > 0$ there exists $\lim_{x\to\infty} F|_{A_i}(x,x) =: \alpha_i$.*
If $\sum_{i\in\mathbb{N}} F(x,x)\chi_{A_i}(x)$ is uniformly convergent with respect to $x \in X$ (to F), then $L_o(F)$ exists and is equal to $\sum_{i=1}^{\infty} L_o(A_i)\alpha_i$ (where α_i can be any real number if $L_o(A_i) = 0$).

Proof. If definitively $A_i = \emptyset$, then the statement follows by induction on n using Theorem A.1 and Proposition 4.2.
Let us consider the general case. If $L_o(A_i) > 0$ then $|A_i| = +\infty$ and then ∞ is an accumulation point of A_i in $\overline{X}$. Since $A_i \cup \{\infty\}$ with the induced topology from $\overline{X}$ is homeomorphic to the one point compactification of A_i (with the induced topology from X), then it is possible to apply Theorem A.1 to $F|_{A_i}$ obtaining $F|_{A_i} \in \mathcal{D}(L_o)$. By Proposition 4.2 we have that $\chi_{A_i}F \in \mathcal{D}(A_i)$ and then $L_o(\sum_{i=1}^{n} \chi_{A_i}F) = \sum_{i=1}^{n} L_o(A_i)\alpha_i$. Using Proposition A.3 we have the conclusion. □

We remark that even though sets of measure zero have no influence on the resulting limit on the average of the function F their presence may change the return probabilities and hence the function F that we average. This is the main difficulty in the second approach.

Anyway, under certain regularity conditions we can gain information on the whole graph from the knowledge of what happens on its subgraph (regarded as an independent graph). With the next theorem we give a sufficient condition for the simple random walk on a general graph to be TOA when one of its subgraphs is locally transient (that is the simple random walk on it is locally transient). The proof is an application of Theorem 3.10.

Theorem 4.6. *Let $(A, E(A))$ be a subgraph of X such that $\sup L_o(A) > 0$. Suppose that there exists $x_0 \in A$ such that for every vertex $y \in A$ there exists an injective map $\gamma_y : A \to A$ such that (i) $\gamma_y(x_0) = y$ and (ii) for any $w, z \in A$, $(w,z) \in E(A)$ implies $(\gamma_y(w), \gamma_y(z)) \in E(A)$. If the simple random walk on $(A, E(A))$ is transient then the simple random walk on $(X, E(X))$ is* TOA.

The proof will be given in the general case, see Theorem 6.4.

We observe that the condition on A in the previous statement is a requirement of "self-similarity" of A (take for instance Cayley graphs).

Corollary 4.7. *Let $(G, E(G))$ be a Cayley graph and $A \subseteq G$ such that (i) the group identity $e \in A$, (ii) for any $x, y \in A$ we have that $xy \in A$ and (iii) the simple random walk on $(A, E(A))$ is transient. If $(X, E(X))$ is a locally finite graph which contains $(A, E(A))$ as a subgraph and $\sup L_o(A) > 0$ then the simple random walk on $(X, E(X))$ is* TOA.

We observe that Theorem 4.6 and Corollary 4.7 hold for any λ (indeed once the hypothesis $\sup L_\lambda(A) > 0$ is satisfied, λ plays no role in the proof).

The last result of this section deals with knowledge of random walks on subgraphs and the thermodynamical limit on the average. Before stating it we need a technical lemma

and a definition.

Lemma 4.8. *Let $(X, E(X))$ be a graph with bounded geometry and C a measurable subset of X such that $L_o(C) = 0$. Let $X_n := \{x \in X : d(x, C) \leq n\}$, then X_n is measurable and $L_o(X_n) = 0$ for every $n \in \mathbb{N}$.*

Proof. We note that for every $n, r \in \mathbb{N}$, $X_n \cap B(o, r) \subseteq \bigcup_{x \in C \cap B(o, n+r)} B(x, n)$ and by hypotheses,

$$|B(o, r+n)|/|B(o, r)| \leq |B(x, n)| \leq M,$$

where $M = \sup_{x \in X} |B(x, n)|$. Then

$$\frac{|X_n \cap B(o, r)|}{|B(o, r)|} \leq M \frac{|C \cap B(o, r+n)|}{|B(o, r)|} \leq M^2 \frac{|C \cap B(o, r+n)|}{|B(o, r+n)|} \stackrel{r \to +\infty}{\longrightarrow} 0.$$

□

Our purpose is now to consider a subgraph $(A, E(A))$ as an independent graph, nevertheless the random walk we study on it should be closely related to the random walk (X, P) that we suppose adapted to $(X, E(X))$ (think for instance of the simple random walk): this is the aim of the following definition.

Definition 4.9. *Let $(A, E(A))$ be a subgraph on $(X, E(X))$ and let (X, P) be a random walk on $(X, E(X))$. A random walk (A, P_A) is called* induced random walk *if for every $x \in A \setminus \partial A$ and every $y \in A$ we have that $p(x, y) = p_A(x, y)$.*

We note that in general the induced random walk is not uniquely determined, but if $n \in \mathbb{N}$ and $x \in A$ are such that $d(x, \partial A) \geq n$ then

$$f^{(n)}(x, x) = f_A^{(n)}(x, x), \qquad p^{(n)}(x, x) = p_A^{(n)}(x, x). \tag{6}$$

In the next theorem we deal with a graph which is partitioned in two subgraphs with known properties. We require that a subset A is *convex* with respect to a vertex $o \in A$, that is that for every $x \in A$ at least one geodesic path from o to x lies in A (hence we are sure that $d_A(o, x) = d_X(o, x)$ and we denote this distance simply by d).

Theorem 4.10. *Let $(X, E(X))$ be an infinite graph with bounded geometry, and let $(A, E(A))$ and $(B, E(B))$ be two subgraphs such that $\{o\} = A \cap B$, $X = A \cup B$ and A, B are both convex with respect to o. Moreover suppose that $L_o(A) > 0$ and $L_o(\partial A) = 0$. Let P be a stochastic matrix representing a random walk on X (adapted to $(X, E(X))$) and let us consider two induced random walks (represented by P_A and P_B) on the subgraphs $(A, E(A))$ and $(B, E(B))$. Under the previous hypotheses we have that*

(i) any two of the following assertion imply the remaining one:

(i.a) (X, P) is L_o-thermodynamically classifiable;

(i.b) (A, P_A) is L_o-thermodynamically classifiable;

(i.c) (B, P_B) is L_o-thermodynamically classifiable;

(ii) if two of the assertions in (i) hold then (A, P_A) $\mathrm{TOA_t}$ *implies (X, P)* $\mathrm{TOA_t}$*;*

(iii) if two of the assertions in (i) hold and $L_o(A) < 1$ then (X, P) is $\mathrm{ROA_t}$ *if and only if (A, P_A) and (B, P_B) are both* $\mathrm{ROA_t}$*.*

Proof. (i) It follows easily by Proposition 4.2 and by the equation (7) below.

(ii) Let F and F_A be the generating functions (depending on $x \in X$ and $z \in [0,1)$) of the hitting probabilities associated to P and P_A respectively. By the hypotheses $F(z) \in \mathcal{D}(L_o)$ and $F_A(z) \in \mathcal{D}(L_o^A)$ for all $z \in (\varepsilon, 1)$, for some $\varepsilon \in (0,1)$. By equation (6) and Lemma 4.8 we can apply Proposition B.3 to F_A and $F_{|A}$ obtaining that $F_{|A}(z) \in \mathcal{D}(L_o^A)$ and $L_o^A(F_{|A}(z)) = L_o^A(F_A(z))$ for all $z \in (\varepsilon, 1)$.

From Proposition 4.2 we have that $\chi_A F \in \mathcal{D}(L_o)$, moreover $L_o^A(F_A(z)) \equiv L_o^A(F_{|A}) = L_o(\chi_A F(z))/L_o(A)$. We note that

$$L_o(F(z)) = L_o(\chi_A F(z)) + L_o(\chi_B F(z)) - L_o(\chi_{\{o\}} F(z)) \leq L_o^A(F_A(z)) L_o(A) + (1 - L_o(A)), \tag{7}$$

hence if $\lim_{z \to 1^-} L_o^A(F_A(z)) < 1$ we obviously have $\lim_{z \to 1^-} L_o(F(z)) < 1$.

(iii) The *only if* part is a consequence of (ii). As for the *if* part, let us define P_* by

$$p_*(x,y) = \begin{cases} p_A(x,y) & \text{if } x, y \in A,\ x \neq o \\ p_B(x,y) & \text{if } x, y \in B,\ x \neq o \\ (1/2) p_A(o,y) & \text{if } x = o,\ y \in A \\ (1/2) p_B(x,o) & \text{if } x = o,\ y \in B, \\ 0 & \text{otherwise} \end{cases}$$

which is clearly a stochastic matrix satisfying $p_*(x,y) = p(x,y)$ for every $x \notin \partial A \cup \partial B$. Since $\deg(\cdot)$ is bounded, we have that $L_o(\partial A) = 0$ if and only if $L_o(\partial B) = 0$; moreover if we have $x \in X$ and $n \in \mathbb{N}$ such that $n \leq d(x, \partial A \cup \partial B)$ then $f_*^{(n)}(x,x) = f^{(n)}(x,x)$. Whence, using again Proposition B.3 (with $C := \partial A \cup \partial B$ and X_n defined as in Lemma 4.8) and Proposition 4.2, we show that $L_o(F(z)) = L_o(F_*(z)) = L_o^A(F_A(z)) L_o(A) + L_o^B(F_B(z)) L_o(B) \stackrel{z \to 1^-}{\longrightarrow} L_o(A) + L_o(B) = 1$, whence the proof is complete. $\square$

The previous theorem is different from those in [1] since here a subgraph A is regarded as an independent graph with an induced random walk. In [1], one is supposed to study the generating function F of (X,P) to classify the random walk; in our approach one can study independently two (hopefully) simpler random walk P_A and P_B (on A and B respectively) and then the classification of the main random walk can be inferred.

5. Averages over increasing sequences of subsets

In this section we deal with the classification on the average of a random walk when the average is taken over a family $\mathcal{F}$ of subsets of X. More precisely, we consider $\mathcal{F} = \{B_n\}_n$ an increasing sequence of finite subsets of X such that $\bigcup_n B_n = X$ (we call $\mathcal{F}$ an *increasing covering family* or *ICF*), and denote by $L_{\mathcal{F}}$ the corresponding limit on the average.

Clearly, two families of subsets $\mathcal{F}_1 = \{B_n\}_n$ and $\mathcal{F}_2 = \{C_n\}_n$ may give the same classification on the average of a random walk. The following proposition provides a sufficient condition for this to happen.

Proposition 5.1. *Given two increasing covering families $\mathcal{F}_1 = \{B_n\}_n$ and $\mathcal{F}_2 = \{C_n\}_n$ in X, such that*

(i) there exist a divergent sequence $\{i_n\}_n$ of natural numbers and $K > 0$ such that $B_{i_n} \supseteq C_n$ and $|B_{i_n}|/|C_n| \leq K$, for every n;

(ii) there exist a divergent sequence $\{j_n\}_n$ of natural numbers and $K' > 0$ such that $C_{j_n} \supseteq B_n$ and $|C_{j_n}|/|B_n| \leq K'$, for every n;

then $L_{\mathcal{F}_1}$ and $L_{\mathcal{F}_2}$ induce the same classification on the average of any random walk.

Proof. These conditions are equivalent to the ones in Proposition 6.1 (i). □

We already observed that the family of L_o-measurable sets is not an algebra. We now prove it for the more general case of $L_{\mathcal{F}}$-measurable sets (where $\mathcal{F}$ is an ICF).

Proposition 5.2. *Let $\mathcal{F} = \{B_n\}_n$ be an ICF of X. Then the class of $L_{\mathcal{F}}$-measurable subsets is not an algebra; in particular there exist A, B $L_{\mathcal{F}}$-measurable subsets of X such that $A \cap B$ is not $L_{\mathcal{F}}$-measurable.*

Proof. Let us define, for every $n \in \mathbb{N}$, $m_n := |B_n|$; let $\{A_k, C_k\}_{k\in\mathbb{N}}$ be a family of subsets of X such that for every $k \in \mathbb{N}$, $\{A_k, C_k\}$ is a partition of $S_k = B_k \setminus B_{k-1}$ with the following two properties

$$|A_k| - |C_k| \in \{0, \pm 1\}, \quad \forall k \in \mathbb{N},$$
$$|\cup_{i=0}^k A_i| - |\cup_{i=0}^k C_i| \in \{0, \pm 1\}, \quad \forall k \in \mathbb{N},$$

Let us define for every $k \in \mathbb{N}$, $a_k := |\cup_{i=0}^k A_i|$, $c_k := |\cup_{i=0}^k C_i|$; since X is infinite, we can choose an increasing sequence of natural numbers $\{k_n\}_n$ such that $m_{k_{n+1}}/m_{k_n} \geq 4$. It is easy to note that by our hypotheses, for every $n, i \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{2} - \frac{1}{m_n} \leq \frac{a_n}{m_n} \leq \frac{1}{2} + \frac{1}{m_n} \\ \frac{1}{2} - \frac{1}{m_n} \leq \frac{c_n}{m_n} \leq \frac{1}{2} + \frac{1}{m_n} \\ \frac{m_i - 2}{m_n + 2} \leq \frac{a_i}{a_n} \leq \frac{m_i + 2}{m_n - 2}. \end{aligned} \tag{8}$$

We finally define the two sets

$$A := \bigcup_{i=0}^{\infty} A_i, \qquad B := \bigcup_{i=0}^{\infty} \left(\bigcup_{j=k_{2i}+1}^{k_{2i+1}} A_j \cup \bigcup_{j=k_{2i+1}+1}^{k_{2i+2}} C_j \right);$$

by equation (8) (since $|A \cap B_n| = a_n$) we have that A is measurable and $L_{\mathcal{F}}(A) = 1/2$; similarly $||B \cap B_n| - c_n| \leq 1 + |\{i \in \mathbb{N} : k_i < n\}|$. Since $m_{k_{n+1}}/m_{k_n} \geq 4$ we have that $\lim_{n\to+\infty} |\{i \in \mathbb{N} : k_i < n\}|/m_n = 0$ (observe that if $|\{i \in \mathbb{N} : k_i < n\}| = j$ then $m_n \geq 4^j$), then by equation (8) we obtain that B is also measurable and $L_{\mathcal{F}}(B) = 1/2$. Moreover $A \cap B = \bigcup_{i=0}^{\infty} \bigcup_{j=k_{2i}+1}^{k_{2i+1}} A_j$, hence if n is odd

$$\frac{|A \cap B \cap B_{k_n}|}{|B_{k_n}|} \geq \frac{a_{k_n}}{m_{k_n}} \left(1 - \frac{a_{k_{n-1}}}{a_{k_n}}\right) \overset{n\to+\infty}{\longrightarrow} \frac{1}{4};$$

similarly when n is even (since $m_{k_{n+1}}/m_{k_n} \geq 4$ and using equation (8))

$$\frac{|A \cap B \cap B_{k_n}|}{|B_{k_n}|} \leq \frac{a_{k_{n-1}}}{m_{k_n}}.$$

Since $\liminf_{n\to+\infty} \frac{a_{k_{n-1}}}{m_{k_{n-1}}} \frac{m_{k_{n-1}}}{m_{k_n}} \leq 1/8$, we have that $\inf L_{\mathcal{F}}(A \cap B) \leq 1/8$ meanwhile $1/4 \leq \sup L_{\mathcal{F}}(A \cap B)$ which implies that $A \cap B$ is not measurable. $\square$

We give two examples of averages over subsets which are not balls. The first one shows that with two different ICFs the classification of a random walk can be different; it is also an example of a random walk which is TOA even if $L_o(G) = +\infty$ (recall Proposition 3.9). The second one is an example of classification on the average with an ICF which appears natural and with respect to which the random walk is ROA and $\mathrm{TOA_t}$.

Example 5.3. Let X be the graph obtained from $\mathbb{Z}^3$ by deleting all horizontal edges joining vertices with positive height (compare with [1] where this graph is an example of mixed $\mathrm{TOA_t}$ and see Figure 6): we call X_+ the set of vertices with (strictly) positive height and $X_- = X_+^c$. The simple random walk on X is locally transient; using Theorem 2.5 one can construct a finite energy flow u defined on $E(\mathbb{Z}^3)$ from the origin o to ∞ with input 1. By Corollary 4.7 we have that the simple random walk is TOA (more generally it is TOA with respect to any λ such that $\sup L_\lambda(X_-) > 0$). Note that $L_o(G) = \infty$ (more generally given a general limit in the average $L_\lambda(G) = +\infty$ if and only if $L_\lambda(X_-) > 0$).

The same graph is $\mathrm{ROA_t}$ (thus ROA) with respect to the following ICF: $\mathcal{F} = \{B_n = (B(o,2^n) \cap X_+) \cup (B(o,n) \cap X_-)\}_n$. In this case X_- has $L_{\mathcal{F}}$-measure zero and by Lemma 4.8, for every n, $f^{(n)}(x,x) = f^{(n)}_{\mathbb{Z}}$ outside a set of $L_{\mathcal{F}}$-measure zero. Thus by Theorem A.1 every $f^{(n)}$ has $L_{\mathcal{F}}$ limit equal to $f^{(n)}_{\mathbb{Z}}$ and the graph is $\mathrm{ROA_t}$ (Theorem B.1(a.1)).

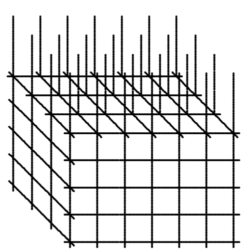
Figure 5

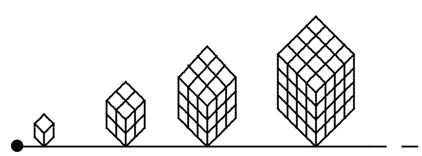
Figure 6

The following example was suggested by D. Cassi, R. Burioni and A. Vezzani.

Example 5.4. Let X be the graph obtained by attaching at each vertex i of $\mathbb{N}$ a cube lattice C_i of side n_i by one of its corners (in Figure 7 $n_i = i$). Suppose that n_i diverges. Then the simple random walk on X is locally recurrent (by an application of Theorem 2.5), hence it is also ROA with respect to any λ.

Consider the following ICF: $\mathcal{F} = \{\bigcup_{i=1}^n C_i\}_n$. The simple random walk on X is $\mathrm{TOA_t}$ with respect to the limit on the average $L_{\mathcal{F}}$. Indeed for each $k \in \mathbb{N}$, the set X_k obtained removing from X all the vertices at distance k from the surface of the cubes has $L_{\mathcal{F}}$-measure equal to 1 and if $x \in X_k$, $f^{(k)}(x,x) = f^{(k)}_{\mathbb{Z}^3}$. The thesis is an easy consequence of Theorem A.1 and Theorem B.1(a.1). Note that since the classification on the average and

the thermodynamical one are different this also provides an example of a random walk for which $F(x,x)$ is not totally convergent (Proposition 6.2(iv)).

6. The general case

In this section we consider a general sequence $\lambda = \{\lambda_n\}_n$ of probability measures on X. We already stated many results for the average over balls which hold also for a general λ: Proposition 3.5, Proposition 3.9, Theorem 3.10, Proposition 4.2, Corollary 4.4, Theorem 4.6 and Corollary 4.7.

We now prove a result which we already used in the previous sections: it is a comparison between classifications with two different limits on the average (see Propositions 3.3 and 5.1).

Proposition 6.1. *Let $\lambda = \{\lambda_n\}_n$, $\eta = \{\eta_n\}_n$ two sequences of probability measures on X. Let us consider the following assertions:*

(i) there exist two divergent sequences $\{i_n\}_n$ and $\{j_n\}_n$ of natural numbers, and two positive constants C, K such that $C\lambda_{i_n}(x) \geq \eta_n(x)$ and $K\eta_{j_n}(x) \geq \lambda_n(x)$ for every n and x;

(ii) for every $A \subseteq X$, $A \in \mathcal{D}(L_\lambda)$, $L_\lambda(A) = 1$ if and only if $A \in \mathcal{D}(L_\eta)$, $L_\eta(A) = 1$;

(iii) a random walk (X,P) is ROA *(respectively* TOA*) with respect to λ if and only if it is* ROA *(respectively* TOA*) with respect to η.*

Then the following chain of implications holds: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii) It is easy (recall also Remark 2.4).

(ii) $\Rightarrow$ (iii) It is true because of the equivalence between (ii) and (iii) in Proposition A.2. $\square$

We state and prove for the general case some of the results quoted in Sections 3 and 4.

Proposition 6.2. *Let (X,P) be a random walk and let ∞ be the point added to X in order to construct its one point compactification.*

*(i) If there exists $A \subseteq X$ measurable, such that $L_\lambda(A) = 1$ and $\lim_{\substack{x\to\infty \\ x \in A}} F(x,x) = \alpha$ then $L_\lambda(F)$ exists, and is equal to α. Thus the random walk is λ-*TOA *(respectively λ-*ROA*) if and only if $\alpha < 1$ (respectively $\alpha = 1$);*

*(ii) if (X,P) is (locally) recurrent then $L_\lambda(F)$ exists, is equal to 1 and the random walk is λ-*ROA*;*

*(iii) if (X,P) is λ-$\mathrm{ROA_t}$ then $L_\lambda(F)$ exists, is equal to 1 and the random walk is λ-*ROA*;*

*(iv) if the series $F(x,x)$ is totally convergent (with respect to $x \in X$) and (X,P) is λ-$\mathrm{TOA_t}$ then $L_\lambda(F)$ exists, it is less than 1 and the random walk is λ-*TOA*;*

*(v) (X,P) is λ-*ROA *$\Longleftrightarrow$ for every $\varepsilon > 0$ the set $\{x : F(x,x) \geq 1-\varepsilon\}$ is measurable with measure 1;*

(vi) *if there exists $A \subseteq X$ measurable, such that $L_\lambda(A) = 1$ and $\lim_{\substack{x\to\infty \\ x\in A}} F(x,x) = 1$ then (X,P) is* λ-ROA. *Moreover, if λ is regular the converse holds;*

(vii) *(X,P) is* λ-TOA $\iff$ *there exists $A \subseteq X$ such that* $\sup L_\lambda(A) > 0$ *and* $\sup_A F(x,x) < 1$.

Proof.

(i) It is an easy consequence of Proposition 4.5.

(ii) It follows trivially from (i).

(iii) It is a consequence of Remark A.4 since $z \mapsto F(x,x|z)$ is a non decreasing function on $[0,1]$ bounded from above by 1.

(iv) It follows by Theorem B.1(c).

(v) and (vii) From the relation $0 \le \inf L_\lambda(F) \le 1$ we have that (X,P) is λ-TOA if and only if it is not λ-ROA; Proposition A.2 yields the conclusion.

(vi) It is a consequence of Proposition A.2. □

Theorem 6.3. *Let (X,P) be a reversible random walk, with reversibility measure m satisfying* $\inf m(x) > 0$, $\sup m(x) < +\infty$ *(in particular this condition is satisfied by the simple random walk on a graph with bounded geometry). Then TFAE:*

(a) *the random walk is* λ-TOA*;*

(b) *there exists $A \subseteq X$ such that* $\sup L_\lambda(A) > 0$*, there is a finite energy flow u^x from x to ∞ with non-zero input i_0 for every $x \in A$ and* $\sup_{x\in A} < u^x, u^x > < +\infty$*;*

(c) *there exists $A \subseteq X$ such that* $\sup L_\lambda(A) > 0$ *and* $\inf_{x\in A} \mathrm{cap}(x) > 0$.

Proof. First note that the interesting case is (X,P) (locally) transient. We therefore restrict to this particular case.

(a) $\Rightarrow$ (b) Recall that $u^x = -\frac{i_0}{m(x)} \nabla G(\cdot, x)$ is a finite energy flow from x to ∞ with input i_0 and energy

$$< u^x, u^x > = \frac{i_0^2}{m(x)} G(x,x), \tag{9}$$

(where ∇ denotes the difference operator, see [3]). But by Proposition 6.2(vii) the network is λ-TOA if and only if there exists $\alpha < 1$, $A \subseteq X$ such that $\inf L_\lambda(A) > 0$ and $F(x,x) < \alpha < 1$ for every $x \in A$. Since $G(x,x) = 1/(1 - F(x,x))$ this is equivalent to $\sup_{x\in A} G(x,x) < +\infty$. By equation (9) and our hypotheses on the reversibility measure, this implies (b).

(b) $\Rightarrow$ (c) This is an obvious consequence of $\mathrm{cap}(x) \ge 1/ < u^x, u^x >$ (see for instance [3]).

(c) $\Rightarrow$ (a) This follows from $G(x,x) \le m(x)/\mathrm{cap}(x)$ for every $x \in A$, and from our hypotheses on m. □

Theorem 6.4. *Let $(A, E(A))$ be a subgraph of X such that* $\sup L_\lambda(A) > 0$. *Suppose that there exists $x_0 \in A$ such that for every vertex $y \in A$ there exists an injective map $\gamma_y : A \to A$ such that (i) $\gamma_y(x_0) = y$ and (ii) for any $w, z \in A$, $(w,z) \in E(A)$ implies*

$(\gamma_y(w), \gamma_y(z)) \in E(A)$. *If the simple random walk on* $(A, E(A))$ *is transient then the simple random walk on* $(X, E(X))$ *is* λ-TOA.

Proof. Let us consider $(A, E(A))$ with the edge orientation induced by X. Let u be a flow on $(A, E(A))$ with finite energy starting from x_0 to infinity with input 1. Given any simple random walk, the conductance is the characteristic function of the edges, hence for any $y \in A$ it is easy to show that the following equation

$$u_y(a,b) := \begin{cases} \varepsilon_{\gamma_y}(a,b) u({\gamma_y}^{-1}(a), {\gamma_y}^{-1}(b)) & \text{if } (a,b) \in E(A); \\ 0 & \text{if } (a,b) \notin E(A); \end{cases} \qquad \forall (a,b) \in E(X)$$

(where $\varepsilon_{\gamma_y}(a,b)$ is equal to $+1$ or -1 according to $(\gamma_y(a), \gamma_y(b))^+ = (x,y)^+$ or not) define a finite energy flow u_y on $(X, E(X))$ starting from y to ∞ with input one. Apply now Theorem 6.3. □

Proposition 6.5. *Let* (X, P) *be an irreducible random walk. If* $F(\cdot, \cdot | z) \in \mathcal{D}(L_\lambda)$ *for every* $z \in (\varepsilon, 1)$ *for some* $\varepsilon \in (0,1)$ *and* $\lim_{z \to 1^-} L_\lambda(F(z)) = 1$ *(respectively* $L_\lambda(F) = 1$*) then* $\lim_{z \to 1^-} \inf L_\lambda(G(z)) = +\infty$ *(respectively* $L_\lambda(G) = +\infty$*).*

Proof. From equation $G(x,x|z) = 1/(1 - F(x,x|z))$ (see [3]) we have that for all $x \in X$ and for all $z \in \mathbb{R}$, $|z| < 1$, $G(x,x|z) = \varphi(F(x,x|z))$, where $\varphi(t) := 1/(1-t)$. By Jensen's inequality

$$\varphi\left(\sum_{x \in X} F(x,x|z) \lambda_n(x) \right) \le \sum_{x \in X} G(x,x|z) \lambda_n(x).$$

If we take the limit as n goes to infinity of both sides of the previous equation, taking into account the continuity of φ,

$$\begin{aligned} \varphi(L_\lambda(F(z))) &= \lim_{n \to +\infty} \varphi\left(\sum_{x \in X} F(x,x|z) \lambda_n(x) \right) \\ &\le \liminf_{n \to +\infty} \sum_{x \in X} G(x,x|z) \lambda_n(x) =: \inf L_\lambda(G(z)); \end{aligned}$$

hence

$$\lim_{z \to 1^-} \inf L_\lambda(G(z)) \ge \liminf_{z \to 1^-} \varphi(L_\lambda(F(z))) = +\infty.$$

The case $L_\lambda(F) = 1$ is completely analogous (please note that it could happen that $\sum_{x \in X} G(x,x) \lambda_n(x) = +\infty$ for some $n \in \mathbb{N}$). □

Appendix A: limits on the average of general functions

In this appendix we consider a very general setting: if not otherwise stated X is a countable set, λ is a sequence of probability measures on X and f is a function defined on

X. We look for sufficient conditions on f for the existence of $L_\lambda(f)$ and we study what can be said about its value.

First we observe that if λ is regular, values taken by f on finite subsets (which have L_λ-measure zero) do not influence neither the existence nor the value of $L_\lambda(f)$. In some sense only values $f(x)$ for x tending to ∞ matter, where ∞ is the point at infinity of the one point compactification of X.

Theorem A.1. *Let λ be regular, then for any real valued function f,*

$$\liminf_{x\to\infty} f(x) \le \inf L_\lambda(f) \le \sup L_\lambda(f) \le \limsup_{x\to\infty} f(x).$$

In particular if there exists $\lim_{x\to\infty} f = \alpha$ then $f \in \mathcal{D}(L_\lambda)$ and $L_\lambda(f) = \alpha$. Moreover, if f is bounded and $A \subseteq X$ such that $L_\lambda(A) = 1$, then

$$\liminf_{\substack{x\to\infty\\ x\in A}} f(x) \le \inf L_\lambda(f) \le \sup L_\lambda(f) \le \limsup_{\substack{x\to\infty\\ x\in A}} f(x).$$

Proof. We deal only with the first inequality. Suppose that $\liminf_{x\to\infty} f(x) = \alpha \neq -\infty$ (otherwise there is nothing to prove). Since

$$\liminf_{x\to\infty} f(x) \equiv \sup_{S\subset X:|S|<+\infty} \inf_{x\notin S} f(x),$$

for every $\varepsilon > 0$ there exists a finite subset S such that for every $x \notin S$, $f(x) > \alpha - \varepsilon$ and hence

$$\sum_{x\in X} f(x)\lambda_n(x) \ge \sum_{x\in S} f(x)\lambda_n(x) + (1-\lambda_n(S))(\alpha-\varepsilon) \overset{n\to+\infty}{\longrightarrow} \alpha - \varepsilon,$$

whence $\alpha \le \inf L_\lambda(f)$. The rest of the proof is analogous. $\square$

We note that the previous result means that, for λ regular, if $f(x)$ converges ("almost surely") when x goes to infinity then $f \in \mathcal{D}(L_\lambda)$ and $L_\lambda(f)$ does not depend on the choice of λ (only the notion of "almost surely" does).

Moreover, only the topological (discrete) structure of X is involved; if X has a graph structure then the topology is obviously independent of the choice of the edge set (the topology is always the discrete one).

If we restrict to bounded functions, we get a stronger result.

Proposition A.2. *Let $f : X \to \mathbb{R}$ be such that $N \le f(x) \le M$, for every $x \in X$). Then TFAE:*

(i) $f \in \mathcal{D}(L_\lambda)$ *and* $L_\lambda(f) = M$ *(respectively* $L_\lambda(f) = N$*);*

(ii) $\inf L_\lambda(f) = M$ *(respectively* $\sup L_\lambda(f) = N$*);*

(iii) $\forall \varepsilon > 0$, $L_\lambda(\{x : f(x) > M - \varepsilon\}) = 1$ *(respectively* $L_\lambda(\{x : f(x) < N + \varepsilon\}) = 1$*);*

(iv) $\forall \varepsilon > 0$, $L_\lambda(\{x : f(x) \le M - \varepsilon\}) = 0$ *(respectively* $L_\lambda(\{x : f(x) \ge N + \varepsilon\}) = 0$*).*

Moreover if λ is regular then

(v) there exists a measurable set A with measure 1 such that $\lim_{\substack{x\to+\infty \\ x\in A}} f(x) = M$ (respectively $\lim_{\substack{x\to+\infty \\ x\in A}} f(x) = N$)

is equivalent to each of the previous ones.

Proof. We consider the case involving the superior limit M (the other one is completely analogous). Let us define $F_\varepsilon^+ := \{x : f(x) > M - \varepsilon\}$ and $F_\varepsilon^- := X \setminus F_\varepsilon^+$.

$(i) \Leftrightarrow (ii)$ and $(iii) \Leftrightarrow (iv)$ are trivial.

$(i) \Rightarrow (iii)$. For every $n \in \mathbb{N}$, $\varepsilon > 0$, $\sum_{x\in X} f(x)\lambda_n(x) \le M\lambda_n(F_\varepsilon^+) + (M-\varepsilon)\lambda_n(F_\varepsilon^-)$, whence

$$0 \le \sup L_\lambda(F_\varepsilon^-) \le \frac{1}{\varepsilon}(M - \inf L_\lambda(f)) \le \frac{1}{\varepsilon}(M - L_\lambda(f)) = 0.$$

$(iii) \Rightarrow (i)$. By Remark 2.2, $f\chi_{F_\varepsilon^-} \in \mathcal{D}(L_\lambda)$ and $L_\lambda(f\chi_{F_\varepsilon^-}) = 0$ for every $\varepsilon > 0$. Whence for every $\varepsilon > 0$

$$M \ge \sup L_\lambda(f) \ge \inf L_\lambda(f) = \inf L_\lambda(f\chi_{F_\varepsilon^+}) \ge (M-\varepsilon)L_\lambda(F_\varepsilon^+) = M - \varepsilon,$$

which easily implies (i).

$(v) \Rightarrow (iii)$. Let $n \in \mathbb{N}$ and $A_n := \{x \in X : f(x) > M - 1/n\}$. Since $A \setminus A_n$ is finite and $A_n^c = (A \setminus A_n) \cup (A^c \setminus A_n)$, then $L_\lambda(A_n^c) = 0$.

$(i) \Rightarrow (v)$. Let $\{B_n\}_n$ be an increasing sequence of finite subsets of X such that $\cup_{n\in\mathbb{N}} B_n = X$ (that is $\{B_n\}_n$ is a basis for the set of neighbours of ∞). Let, for any $n \in \mathbb{N}$, A_n defined as in the previous point (A_n is non-empty since $L_\lambda(f) = M$). Let us construct recursively two increasing sequences $\{m_i\}_i$ and $\{n_i\}_i$ with values in $\mathbb{N}$ satisfying

$$\begin{aligned} &\lambda_m(A_i) > 1 - 1/i, \qquad \forall m \ge m_i; \\ &\lambda_m(A_i \cap B_{n_i}) > 1 - 1/i, \qquad \forall m : m_i \le m < m_{i+1}. \end{aligned}$$

This is possible since $\lim_{m\to+\infty} \lambda_m(A_i) = 1$ for any $i \in \mathbb{N}$ and since (using Monotone Convergence Theorem) $\lim_{n\to+\infty} \lambda_m(A_i \cap B_n) = \lambda_m(A_i) > 1 - 1/i$ and the set $\{m : m_i \le m < m_{i+1}\}$ is finite. We prove now that $A := \cup_{i=1}^\infty (A_i \cap B_{n_i})$ satisfies the two conditions in (v).

By regularity we have that $L_\lambda(A \setminus B_{n_i}) = 1$ hence $A \setminus B_{n_i} \ne \emptyset$ and ∞ is an accumulation point for A. Moreover if $x \in A \setminus B_{n_i}$ we have that $x \in A_j$ for some $j > i$ and hence $f(x) > M - 1/j > M - 1/i$; this proves that $\lim_{\substack{x\to+\infty \\ x\in A}} f(x) = M$.

If m satisfies $m_i \le m < m_{i+1}$ then

$$\lambda_m(A) \ge \lambda_m(A_i \cap B_{n_i}) > 1 - 1/i$$

whence $\lim_{m\to+\infty} \lambda_m(A) = 1$. $\square$

In the hypotheses of the previous theorem, if $\{\lambda_n\}_n$ is regular and if for some $\varepsilon > 0$, the set $\{x : f(x) > M - \varepsilon\}$ (respectively $\{x : f(x) < N + \varepsilon\}$) is finite, then $L_\lambda(\{x : f(x) = M\}) = 1$ (respectively $L_\lambda(\{x : f(x) = N\}) = 1$).

We show now that $\mathcal{D}(L_\lambda)$ is closed in the uniform convergence topology and that L_λ is continuous.

Proposition A.3. *Let* $(\Gamma, \geq)$ *be a directed, partially ordered set. If* $\{f_\gamma\}_\Gamma \subseteq \mathcal{D}(L_\lambda)$ *is a net with the property that* $\lim_\gamma f_\gamma =: f$ *holds uniformly with respect to* $x \in X$*, then* $f \in \mathcal{D}(L_\lambda)$ *and* $L_\lambda(f) = \lim_\gamma L_\lambda(f_\gamma)$*.*

Proof. It is trivial to note that Theorem 7.11 of [11] holds considering a net of functions instead of a sequence.

Since $f_\gamma - f \to 0$ in $l^\infty(X)$ and $f_\gamma \in \mathcal{D}(L_\lambda)$ for every $\gamma \in \Gamma$, then it is easy to show that $\sum_{x\in X} f(x)\lambda_n(x)$ exists for every $n \in \mathbb{N}$ and

$$g_\gamma(n) := \sum_{x\in X} f_\gamma(x)\lambda_n(x) \xrightarrow{\gamma} \sum_{x\in X} f(x)\lambda_n(x) =: \varphi(n)$$

uniformly with respect to $n \in \mathbb{N}$ (since $\sum_{x\in X}^\infty \lambda_n(x) = 1$).

Since $\lim_{n\to+\infty} g_\gamma(n) =: L_\lambda(f_\gamma)$, we have that $\lim_\gamma L_\lambda(f_\gamma)$ and $L_\lambda(f) := \lim_{n\to+\infty} \varphi(n)$ both exist and they are equal. □

The previous result leads to an alternative proof Theorem A.1: it is enough to observe that if $o \in X$ is fixed and f satisfies $\lim_{x\to\infty} f(x) = \alpha$ then $f_n(x) := (f - \alpha)\cdot \chi_{B(o,n)} + \alpha$ is a sequence of functions uniformly convergent to f, such that $f_n \in \mathcal{D}(L_\lambda)$ and $L_\lambda(f_n) = \alpha$.

Remark A.4. As in the case of the thermodynamical classification, f may depend not only on $x \in X$ but also on $z \in [0, r)$. If $z \mapsto f(x, z)$ is non negative and not decreasing as $z \in (\varepsilon, r)$ (for each fixed x), then

$$\lim_{z\to r^-} \inf L_\lambda(f(\cdot, z)) = \sup_{z\in(\varepsilon,r)} \inf L_\lambda(f(\cdot, z)) \leq \inf L_\lambda(f(\cdot, r)),$$

moreover if $\lim_{z\to r^-} \inf L_\lambda(f(\cdot, z)) = \sup_{x\in X} f(x, r) < +\infty$ then $f(\cdot, r) \in \mathcal{D}(L_\lambda)$ and $L_\lambda(f(\cdot, r)) = \sup_{x\in X} f(x, r) < +\infty$.

Appendix B: limits on the average of power series

In this appendix X and λ are as in the preceding section. We study the limit on the average of families of power series. In particular we search for conditions on $\sum_{n=0}^\infty a_n(x)z^n$, to belong to $\mathcal{D}(L_\lambda)$ for every fixed z in the common domain of convergence; moreover, provided that $L_\lambda(\sum_{n=0}^\infty a_n(x)z^n)$ exists we ask when

$$\lim_{z\to r^-} L_\lambda\left(\sum_{n=0}^\infty a_n(x)z^n\right) = L_\lambda\left(\sum_{n=0}^\infty a_n(x)r^n\right),$$

where $z \in \mathbb{R}$ and $B(0,r)$ is a common domain of convergence.

Theorem B.1. *Let $\sum_{n=0}^{\infty} a_n(x)z^n$ be a family of power series such that $a_n(x) \geq 0$ for every $n \in \mathbb{N}, x \in X$. Suppose that the series $\sum_{n=0}^{\infty} k_n z^n$, where $k_n := \sup_{x \in X} a_n(x)$, has a positive radius of convergence r'. If $r \in (0, r']$ then the following results hold:*

(a) if $\{a_n(\cdot)\}_n \subset \mathcal{D}(L_\lambda)$ then

(a.1) $\sum_{n=0}^{\infty} a_n(\cdot)z^n \in \mathcal{D}(L_\lambda)$ and $L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)z^n) = \sum_{n=0}^{\infty} L_\lambda(a_n)z^n$, for every $z \in \mathbb{C}$ such that $\sum_{n \in \mathbb{N}} k_n |z|^n < +\infty$;

(a.2) $\lim_{z \to r^-} L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)z^n) = \sum_{n=0}^{\infty} L_\lambda(a_n)r^n \leq \inf L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)r^n)$ (the values possibly being infinite),

(a.3) if $\sum_{n=0}^{\infty} k_n r^n < +\infty$ then $\lim_{z \to r^-} L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)z^n) = L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)r^n)$;

(b) if $\sum_{n=0}^{\infty} a_n(\cdot)z^n \in \mathcal{D}(L_\lambda)$ for all $z \in \Gamma$ where Γ is a C^1-circuit with $\sup_{z \in \Gamma} |z| = r_1 < r$, and $d(\Gamma, 0) = r_2 > 0$ then $\{a_n\} \subset \mathcal{D}(L_\lambda)$ and (a.1) holds;

(c) if $\sum_{n=0}^{\infty} k_n r^n < +\infty$, then $\lim_{k \to \infty} L_\lambda(\sum_{n=0}^{\infty} a_n(x)z_k^n) = L_\lambda(\sum_{n=0}^{\infty} a_n(x)r^n)$ provided that $|z_k| < r$, $z_k \to r$ and $\sum_{n=0}^{\infty} a_n(\cdot)z_k^n \in \mathcal{D}(L_\lambda)$ for every k.

Proof. The proof is based on quite classical arguments,hence we just outline it.

(a.1) Exchange the order of summation (using for instance Fubini-Tonelli's Theorem) and apply Bounded Convergence Theorem to the series.

(a.2) It is an easy application of Monotone Convergence Theorem.

(a.3) It follows from (a.1) and (a.2).

(b) Using the Cauchy integral formula and Fubini's Theorem we obtain

$$\sum_{x \in X} a_k(x)\lambda_n(x) = \frac{1}{2\pi i} \int_\Gamma \frac{\sum_{x \in X} \left(\sum_{j=0}^{\infty} a_j(x) z^j \lambda_n(x) \right)}{z^{k+1}} \mathrm{d}z,$$

and simple calculations show that the norm of the integrand is bounded by a constant (uniformly with respect to $n \in \mathbb{N}$), hence Bounded Convergence Theorem implies the result.

(c) It follows by Proposition A.3, since $\lim_{\substack{z \to r \\ |z| < r}} \sum_{n=0}^{\infty} a_n(x)z^n = \sum_{n=0}^{\infty} a_n(x)r^n$ holds uniformly with respect to $x \in X$. □

The meaning of the previous theorem is that for a family of power series satisfying the hypotheses of the theorem, the following assertions are equivalent:

(i) every coefficient is in the domain of L_λ;

(ii) there exists a circuit Γ as in Theorem B.1(b), such that for all $z \in \Gamma$, $\sum_{n=0}^{+\infty} a_n(x)z^n$ is in the domain of L_λ;

(iii) for every $z \in \mathbb{C}$, $|z| < r'$, $\sum_{n=0}^{+\infty} a_n(x)z^n$ is in the domain of L_λ.

By means of the previous theorem we can state and prove a result which we call *identity principle on the average for power series.*

Proposition B.2. *If $w_j(x,z) := \sum_{i=0}^{\infty} a_i^{(j)}(x) z^i$, $j = 1,2$, is a couple of families of power series on X, with non negative coefficients. Suppose that the series $\sum_{i=0}^{\infty} k_i^j z^i$, where $k_i^j := \sup_{x\in X} a_i^{(j)}(x)$, has a positive radius of convergence r_j, $j = 1,2$, and that $w_j(\cdot, z) \in \mathcal{D}(L_\lambda)$ $j = 1,2$, for all $z \in B(0, \min(r_1, r_2))$. Then TFAE*

(i) there exists a subset $E \subseteq B(0, \min(r_1, r_2))$ with an accumulation point x_0 which belongs to the domain $B(0, \min(r_1, r_2))$ such that

$$L_\lambda(w_1(\cdot, z)) = L_\lambda(w_2(\cdot, z)), \quad \forall z \in E;$$

(ii) for every $n \in \mathbb{N}$ and for every $x \in X$ we have that $L_\lambda(a_n^{(1)}) = L_\lambda(a_n^{(2)})$.

Proof. By Theorem B.1(a.1) we have that

$$L_\lambda(w_j(\cdot, z)) = \sum_{i=0}^{\infty} L_\lambda(a_i^{(j)}) z^i, \quad \forall z \in B(o, r_j),\ j = 1,2, \tag{10}$$

then (i) $\Longrightarrow$ (ii) is trivial.
(ii) $\Longrightarrow$ (i) It is a consequence of (10) and Theorems 8.1.2 and 8.1.3 of [12]. □

We can state a similar result which takes also into account the presence of zero-measure sets.

Proposition B.3. *Suppose that $w_i(x,z) := \sum_{j=0}^{\infty} a_j^{(i)}(x) z^j$, $i = 1,2$ satisfy the hypotheses of the previous theorem, and that for every $n \in \mathbb{N}$ there exists a subset $X_n \subset X$ such that $L_\lambda(X_n) = 0$ and $a_n^{(1)}(x) = a_n^{(2)}(x)$ for every $x \notin X_n$. If $z \in \mathbb{C}$, $|z| \in (0, \min(r_1, r_2))$ then*

$$w_1(\cdot, z) \in \mathcal{D}(L_\lambda) \quad \Longleftrightarrow \quad w_2(\cdot, z) \in \mathcal{D}(L_\lambda), \tag{11}$$

and

$$L_\lambda(w_1(\cdot, z)) = L_\lambda(w_2(\cdot, z)). \tag{12}$$

Proof. Given any general family of power series $w(x,z) := \sum_{i=0}^{\infty} a_i(x) z^i$ such that $\sum_{i=0}^{\infty} k_i z^i$ has a positive radius of convergence r ($k_i := \sup_{x\in X} |a_i(x)|$), using Bounded Convergence Theorem it is not difficult to prove that (uniformly with respect to $|z| \le r - \varepsilon$),

$$\lim_{n\to+\infty} \sum_{i=o}^{\infty} \sum_{x\in X_i} a_i(x) z^i \lambda_n(x) = 0.$$

Hence if $|z| < r$ it is obvious that

$$w_j(\cdot, z) \in \mathcal{D}(L_\lambda) \iff \exists \lim_{n\to+\infty} \sum_{i=0}^{\infty} \sum_{x\in X_i^c} a_i^{(j)}(x) z^i \lambda_n(x),$$

$$L_\lambda(w_j(\cdot, z)) = \lim_{n\to+\infty} \sum_{i=0}^{\infty} \sum_{x\in X_i^c} a_i^{(j)}(x) z^i \lambda_n(x),$$

but by our hypotheses $a_i^{(1)}(x) = a_i^{(2)}(x)$ for every $i \in \mathbb{N}$ and for every $x \in X_i^c$, whence the existence and the value of the last limit does not depend on j. $\square$

9. Conclusions

In this paper we proposed a new classification and we showed how is it possible to manage it from a technical point of view. However, following [4], it seems reasonable to consider also the "dual" classification obtained by substitution in definition 2.1 of the $\liminf$ with the $\limsup$. With slight differences, it is possible to prove similar results for this new classification. Here we state two crucial results (Proposition 9.2 and Theorem 9.3) and we give the proof of the former, the proof of the latter being scarcely different from its analogous.

Definition 9.1. *Let (X,P) be a random walk and $\{\lambda_n\}$ a sequence of probability measures on X. The random walk is called λ-Suprecurrent (resp. λ-Suptransient) if and only if* $\sup L_\lambda(F) = 1$ *(resp.* $\sup L_\lambda(F) < 1$*).*

We immediately note that λ-ROAimplies λ-Suprecurrent; moreover a random walk is λ-Suptransient if and only if there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have $\sum_{x\in X} F(x)\lambda_n(x) < 1-\varepsilon$.

The following proposition the analogous of Proposition A.2.

Proposition 9.2. *Let $F : X \to \mathbb{R}$ such that $N \leq F(x) \leq M$ for any $x \in X$; if λ is regular then consider the following assertions*

(i) $\sup L_\lambda(F) = M$ *(resp.* $\inf L_\lambda(F) = N$*);*

(ii) there exists a subset $A \subseteq X$ such that $\sup L_\lambda(A) = 1$ *and* $\lim_{\substack{x\to+\infty\\x\in A}} F(x) = M$ *(resp.* $\lim_{\substack{x\to+\infty\\x\in A}} F(x) = N$*);*

(iii) for any $\varepsilon > 0$, $\sup L_\lambda(\{F > M-\varepsilon\}) = 1$ *(*$\sup L_\lambda(\{F > N+\varepsilon\}) = 1$*). Hence (i) $\Longleftrightarrow$ (iii); moreover if λ is regular (ii) is equivalent to any of the previous ones.*

Proof. (i) $\Longrightarrow$ (ii). Let $\{n_j\}_{j\in\mathbb{N}}$ be such that $\lim_{j\to+\infty}\sum_{x\in X} F(x)\lambda_{n_j}(x) = M$ and let $\eta_j := \lambda_{n_j}$ for any $j \in \mathbb{N}$. Then $L_\eta(F) = M$, hence, by Proposition A.2 (being η regular), there exists $A \subseteq X$ such that $L_\eta(A) = 1$ and $\lim_{\substack{x\to+\infty\\x\in A}} F(x) = M$ (obviously $L_\eta(F) = 1$ implies $\sup L_\lambda(F) = 1$).

(ii) $\Longrightarrow$ (i). Let $\{n_j\}_{j\in\mathbb{N}}$ be such that $\lim_{j\to+\infty}\lambda_{n_j}(A) = 1$ and, given any $\varepsilon > 0$, let $K_\varepsilon \subseteq A$ such that $|A \setminus K_\varepsilon| < +\infty$ and $F|_{K_\varepsilon} > M-\varepsilon$. It is easy to show that

$$\sum_{x\in X} F(x)\lambda_n(x) = \sum_{x\in A} F(x)\lambda_n(x) + \sum_{x\in A^c} F(x)\lambda_n(x) \geq$$
$$\geq \lambda_n(K_\varepsilon)(M-\varepsilon) + \lambda_n(A^c)N;$$

hence

$$M \geq \limsup_{j\to+\infty}\sum_{x\in X} F(x)\lambda_{n_j}(x) \geq \limsup_{j\to+\infty}\lambda_{n_j}(K_\varepsilon)(M-\varepsilon) + \limsup_{j\to+\infty}\lambda_{n_j}(A^c)N = M-\varepsilon$$

since $\lambda_n(K_\varepsilon) = \lambda_n(A) - \lambda_n(A \setminus K_\varepsilon)$ and, by regularity, $\lim_{j\to+\infty} \lambda_{n_j}(A \setminus K_\varepsilon) = 0$.

$(i) \Longrightarrow (iii)$. Let $\eta_k := \lambda_{n_k}$ where $\{n_k\}_k$ is such that

$$\lim_{k\to+\infty} \sum_{x\in X} F(x)\lambda_{n_k}(x) = M,$$

then, for every $\varepsilon > 0$,

$$1 \geq \sup L_\lambda(\{F > M - \varepsilon\}) \geq L_\eta(\{F > M - \varepsilon\}) = 1$$

since we apply Proposition A.2 to η.

$(iii) \Longrightarrow (i)$. We know that there exists a non decreasing sequence $\{n_k\}_k$ such that for any $k > 0$

$$\sum_{x\in X} \chi_{\{F>M-1/k\}}(x)\lambda_n(x) > 1 - \frac{1}{k}, \quad \forall n \geq n_k.$$

If $\eta_k := \lambda_{n_k}$ then, given any $\varepsilon > 0$ and $\delta > 0$, for $k \geq 1/\varepsilon + 1/\delta$ we have that

$$\sum_{x\in X} \chi_{\{F>M-\varepsilon\}}(x)\lambda_n(x) \geq \sum_{x\in X} \chi_{\{F>M-1/k\}}(x)\lambda_n(x) \geq 1 - \frac{1}{k} > 1 - \delta,$$

hence $L_\eta(\{F > 1 - 1/\varepsilon\}) = 1$ and, by Proposition A.2, there exists $A \subseteq X$ such that $L_\eta(A) = 1$ and $\lim_{\substack{x\to+\infty \\ x\in A}} F(x) = M$ which implies $\sup L_\lambda(A) = 1$ □

Condition (ii) of the previous theorem is technically easy and it allows us, in the natural case, to characterize suprecurrence. The following theorem is the analogous of Proposition 6.3: the proof is omitted since it is now straightforward.

Theorem 9.3. *Let (X, P) be a reversible random walk, with reversibility measure m satisfying $\inf m(x) > 0$, $\sup m(x) < +\infty$ (in particular this condition is satisfied by the simple random walk on a graph with bounded geometry). Then TFAE:*

(a) the random walk is λ-Suptransient;

(b) there exists $A \subseteq X$ such that $\inf L_\lambda(A) > 0$, there is a finite energy flow u^x from x to ∞ with non-zero input i_0 for every $x \in A$ and $\sup_{x\in A} < u^x, u^x > < +\infty$;

(c) there exists $A \subseteq X$ such that $\inf L_\lambda(A) > 0$ and $\inf_{x\in A} \mathrm{cap}(x) > 0$.

Acknowledgments

We would like to thank R. Burioni, D. Cassi and A. Vezzani for the stimulating conversations.

The authors acknowledge support from the E.S.I. (Wien, Austria). The first author acknowledges also support from the ARGE Alpen-Adria (Steiermark - Austria).

Bibliography

[1] R. Burioni, D. Cassi, A. Vezzani, *The type-problem on the average for random walks on graphs*, Eur. Phys. J. B **15**, 665 (2000).

[2] G. Pólya, *Über eine Aufgabe der Wahrscheinlichkeitstheorie betreffend die Irrfahrt im Straßennetz*, Math. Ann. **84** (1921), 149-160.

[3] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, **138**, Cambridge Univ. Press, 2000.

[4] D. Cassi, *Local vs Average Behavior on Inhomogeneous Structures: Recurrence on the Average and a Further Extension of Mermin-Wagner Theorem on Graphs*, Phys. Rev. Lett. **76**, 2941 (1996).

[5] D. Cassi, L. Fabbian, *The spherical model on graphs*, J. Phys. A **32**, L93 (1999).

[6] R. Burioni, D. Cassi, *Universal properties of spectral dimension*, Phys. Rev. Lett. **76**, 1091-1093 (1996).

[7] T. Lyons, *A simple criterion for transience of a reversible Markov chain.*, Ann. Prob. **11** (1983), 393-402.

[8] M. Yamasaki, *Discrete potentials on an infinite network*, Mem. Fac. Sci., Shimane Univ. **13** (1979), 31-44.

[9] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley & Sons, New York, (1978).

[10] F. Zucca, *Mean value properties for harmonic functions on graphs and trees*, to appear on Ann. Mat. Pura Appl..

[11] W. Rudin, *Principles of Mathematical Analysis*, Mc Graw-Hill, (1953).

[12] E. Hille, *Analytic Function Theory: vol. I*, Chelsea Publ. co., New York, N.Y. (1959).